On existence and behavior of radial minimizers for the Schrödinger-Poisson-Slater problem.

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Motivation of the problem

Let us start with the Hartree-Fock equations:

$$-\Delta\psi_k + (V(x) - E_k)\psi_k + \psi_k(x)\int_{\mathbb{R}^3} \frac{|\rho(y)|}{|x-y|}\,dy - \sum_{j=1}^{N}\psi_j(x)\int_{\mathbb{R}^3}\frac{\overline{\psi_j(y)}\psi_k(y)}{|x-y|}\,dy = 0,$$

where $\psi_k : \mathbb{R}^3 \to \mathbb{C}$ form an orthogonal set in H^1, $\rho = \frac{1}{N}\sum_{j=1}^{N}|\psi_j|^2$, $V(x)$ is an exterior potential and $E_k \in \mathbb{R}$.

This system appears in Quantum Mechanics in the study of a system of N particles. It has the advantage that is consistent with the Pauli exclusion principle.

Motivation

A surprisingly simple approximation of the exchange term was given by Slater in the form:

$$\sum_{j=1}^{N} \psi_j \int_{\mathbb{R}^3} \frac{\overline{\psi_j(y)}\psi_k(y)}{|x-y|}\, dy \sim C_S \rho^{1/3} \psi_k,$$

where C_s is a positive constant.

J. C. Slater, 1951.

Other power approximations of the exchange term have also been given. Models of this type are referred to as Schrödinger-Poisson-X$^{\alpha}$.

Motivation

We now make $N \to +\infty$; by a mean field approximation, the local density ρ can be estimated as $\rho = |u|^2$, where u is a solution of the problem:

$$-\Delta u(x) + V(x)u(x) + Bu(x) \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|}\, dy = C|u(x)|^{2/3}u(x).$$

This system receives the name of Schrödinger-Poisson-Slater system.

- V. Benci and D. Fortunato, 1998.
- O. Bokanowski and N.J. Mauser, 1999.
- N.J. Mauser, 2001.
- O. Bokanowski, J. L. López and J. Soler, 2003.

The problem

In this talk we are interested in the following version of the Schrödinger-Poisson-Slater problem:

$$\begin{cases} -\Delta u + u + \lambda\phi u = |u|^{p-2}u, \\ -\Delta\phi = u^2. \end{cases} \tag{1}$$

Here $p \in (2,6)$, $\lambda > 0$, $u \in H^1(\mathbb{R}^3)$, $\phi \in D^{1,2}(\mathbb{R}^3)$. We denote by $H^1_r(\mathbb{R}^3)$, $D^{1,2}_r(\mathbb{R}^3)$ the Sobolev spaces of radial functions.

As a first result, it is easy to deduce the existence of positive radial solutions for λ small, by using the Implicit Function Theorem.

The functional

Given $u \in H^1(\mathbb{R}^3)$, we have that $\phi_u = \frac{1}{4\pi|x|} * u^2$ belongs to $D^{1,2}(\mathbb{R}^3)$ and satisfies the equation $-\Delta\phi_u = u^2$.

We define the functional $I_\lambda : H^1(\mathbb{R}^3) \mapsto \mathbb{R}$,

$$I_\lambda(u) = \int \frac{1}{2}(|\nabla u|^2 + u^2) + \frac{\lambda}{4}\phi_u u^2 - \frac{1}{p}|u|^p \, dx = \tag{2}$$

$$\int \left(\frac{1}{2}(|\nabla u(x)|^2 + u(x)^2) + \frac{\lambda}{4} \int \frac{u^2(x)u^2(y)}{4\pi|x-y|} \, dy - \frac{1}{p}|u(x)|^p \right) \, dx.$$

Solutions will be sought as critical points of I.

It is clear that I has a local minimum at zero.

Previous work

This system has been studied recently by many researchers;

- P. D'Avenia, 2002.
- T. D'Aprile and D. Mugnai, 2004.
- O. Sánchez and J. Soler, 2004.
- T. D'Aprile and J. Wei, 2005, 2006 and 2007.
- D. R., 2005 and 2006.
- L. Pisani and G. Siciliano, 2007.
- Z. Wang and H.-S. Zhou, 2007.

Previous work

- H. Kikuchi, 2007 and 2008.
- D. R. and G. Siciliano, 2008.
- A. Ambrosetti and D. R., 2008.
- A. Azzollini and A. Pomponio, 2008.
- I. Ianni and G. Vaira, 2008
- C. Mercuri, 2008.
- L. Zhao and F. Zhao, 2008.

Existence results

$$-\Delta u + u + \lambda u \left(\frac{1}{4\pi|x|} * u^2 \right) = |u|^{p-2}u. \qquad \text{(P)}$$

Theorem

The next diagram sums up the existence results:

	λ small	$\lambda \geq 1/4$
$2 < p < 3$	2 solutions $\quad inf\ I_\lambda\|_{H^1_r(\mathbb{R}^3)} > -\infty$	No solution $\quad inf\ I_\lambda\|_{H^1_r(\mathbb{R}^3)} = 0$
$p = 3$	1 solution $\quad inf\ I_\lambda\|_{H^1_r(\mathbb{R}^3)} = -\infty$	No solution $\quad inf\ I_\lambda\|_{H^1_r(\mathbb{R}^3)} = 0$
$3 < p < 6$	1 solution $\quad inf\ I_\lambda\|_{H^1_r(\mathbb{R}^3)} = -\infty$	1 solution $\quad inf\ I_\lambda\|_{H^1_r(\mathbb{R}^3)} = -\infty$

D.R., 2006.

A glance to the case $p > 3$.

Theorem

For any $\lambda > 0$ there exists a positive critical point of $I = I_\lambda$.

Recall that I has a local minimum at zero. We now claim that $I|_{H_r^1(\mathbb{R}^3)}$ is unbounded below.

A glance to the case $p > 3$.

Theorem

For any $\lambda > 0$ there exists a positive critical point of $I = I_\lambda$.

Recall that I has a local minimum at zero. We now claim that $I|_{H^1_r(\mathbb{R}^3)}$ is unbounded below.

Indeed, given any $u \in H^1_r(\mathbb{R}^3)$, let us compute I along the curve $v : \mathbb{R}^+ \to H^1_r(\mathbb{R}^3)$, $v(t)(x) = t^2 u(t\,x)$.

$$I(v(t)) = \int \frac{t^3}{2}|\nabla u|^2 + \frac{t}{2}u^2 + \lambda \frac{t^3}{4}\phi_u u^2 - \frac{t^{2p-3}}{p}|u|^p \, dx$$

Hence, as $t \to +\infty$, $I(v(t)) \to -\infty$. However, for some values of p, the (PS) property is not known!

A nonexistence result for $p \leq 3$

Theorem

Assume that $p \leq 3$, $\lambda \geq 1/4$, *and let* $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ *be a solution of*

$$\begin{cases} -\Delta u + u + \lambda\phi(x)u = |u|^{p-2}u, \\ -\Delta\phi = u^2. \end{cases}$$

Then, $(u, \phi) = (0, 0)$.

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Multiplying by u and integrating, we have:

Proof

$$\int |\nabla u|^2 + u^2 + \lambda\phi_u\, u^2 - |u|^p = 0. \tag{3}$$

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Multiplying by $|u|$ the equation $-\Delta \phi_u = u^2$, we get:

$$\int |u|^3 = \int -\Delta \phi_u |u| = \int \langle \nabla \phi_u, \nabla |u| \rangle \leq \int |\nabla u|^2 + \frac{1}{4} |\nabla \phi_u|^2.$$

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Inserting the above inequality into (3), we have:

$$\int (\lambda - \frac{1}{4})|\nabla\phi_u|^2 + \underbrace{u^2 + |u|^3 - |u|^p} \leq 0.$$

$\square$

The case $2 < p < 3$

Theorem

Suppose $2 < p < 3$. Then, for any λ positive, there holds:

1. $I_\lambda|_{H^1_r(\mathbb{R}^3)}$ *is w.l.s.c and coercive.*
2. $I_\lambda|_{H^1_r(\mathbb{R}^3)}$ *satisfies the (PS) condition.*

In the proof of coerciveness, we strongly use the fact that the functions are radial! Indeed, we have:

Theorem

Suppose $2 < p < 3$, and λ such that inf $I_\lambda < 0$. Then, inf $I_\lambda = -\infty$.

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Corollary

The functional $I_\lambda|_{H^1_r(\mathbb{R}^3)}$ achieves a global minimum.

Existence of two solutions

Theorem

For $\lambda > 0$ small I_λ has at least two positive critical points.

Choose λ small so that $\inf I_\lambda = I_\lambda(u_1)$ is negative.

- 0 is a local minimum of I_λ.
- $0 \neq u_1$ is a global minimum of I_λ.
- I_λ satisfies the (PS) property.

$\Rightarrow$ I_λ has a mountain-pass critical point u_2.

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We can repeat the whole procedure to the functional:

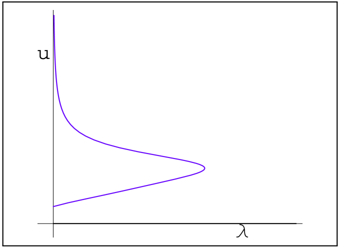
$$I^+(u) = \int \frac{1}{2}(|\nabla u|^2 + u^2) + \frac{\lambda}{4}\phi_u u^2 - \frac{1}{p+1}|u^+|^{p+1}\, dx,$$

and use the maximum principle to show that $u_1 > 0$, $u_2 > 0$.

□

A bifurcation diagram

We also study the bifurcation of the solutions for $p < 3$. One obtains a priori estimates (in H^1 norm) of the radial solutions of (P_λ) for any $\lambda > 0$.



Question: how do minimizers behave asymptotically as $\lambda \to 0^+$?

A singular perturbation result

Theorem

Assume $p \in (2, \frac{18}{7})$ and, for each $s > 0$, define $U_s : \mathbb{R} \to \mathbb{R}$ the unique positive even solution of $-U''(r) + sU(r) = U^p(r)$ in $H^1(\mathbb{R})$.

Then, for λ small there exist radial positive solutions u_λ in the form:

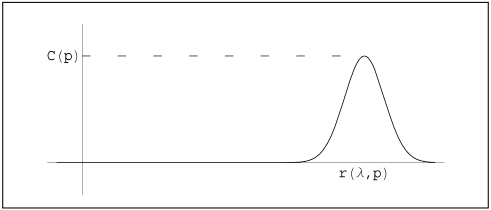
$$u_\lambda(r) \sim U_{a+1}(r - r(\lambda)),$$

where the a, $r(\lambda)$ are given by:

$$a = \frac{8(p-2)}{18-7p}, \quad r(\lambda) = \frac{1}{\lambda} \frac{a}{M(a+1)^{\frac{6-p}{2(p-2)}}}, \quad M = \int_{\mathbb{R}} U_1^2.$$

Moreover, u_λ is a local minimum of the energy functional I_λ, and $I_\lambda(u_\lambda) \to -\infty$ as $\lambda \to 0$.

A perturbation result for λ small



The proof uses a perturbation argument. Consider the manifold:

$$Z = \{U_a(r - \rho), \rho \text{ large}, \ a = f(\lambda\rho)\},$$

where f is a conveniently chosen real function. Z is a manifold of approximate solutions; applying the Lyapunov-Schmidt reduction, we find solutions for $\rho \sim r(\lambda)$.

T. D'Aprile and J. Wei, 2005.

D. R., 2005.

How do minimizers behave for $p \in [18/7, 3)$? Why $p = 18/7$?

The zero mass case

Make the change of variables $v(x) = \varepsilon^{\frac{2}{p-2}} u(\varepsilon x)$, $\varepsilon = \lambda^{\frac{p-2}{4(3-p)}}$, to get:

$$-\Delta v + \varepsilon^2 v + \left(v^2 \star \frac{1}{4\pi|x|} \right) v = |v|^{p-2} v.$$

This motivates the study of the limit problem:

$$-\Delta v + \left(v^2 \star \frac{1}{4\pi|x|} \right) v = |v|^{p-2} v. \tag{4}$$

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It seems quite clear that the right space to study (4) is:

$$E = E(\mathbb{R}^3) = \{ v \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{v^2(x) v^2(y)}{|x-y|} \, dx \, dy < +\infty \}.$$

We denote by E_r its subspace of radial functions.

On the space E

Proposition

E is a uniformly convex Banach space with the norm:

$$\|v\|_E = \left(\int_{\mathbb{R}^3} |\nabla v(x)|^2 \, dx + \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{v^2(x) v^2(y)}{|x-y|} \, dx \, dy \right)^{1/2} \right)^{1/2}.$$

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Moreover, E can be characterized by the space of functions $u \in D^{1,2}(\mathbb{R}^3)$ such that $\phi = \frac{1}{|x|} \star u^2$ also belongs to $D^{1,2}(\mathbb{R}^3)$.

Theorem

$E \subset L^p(\mathbb{R}^3)$ if and only if $p \in [3, 6]$, and the inclusion is continuous.

Observe that E_r is a subset of E, and that symmetric rearrangements do not work properly on E.

Known bounds for the Coulomb energy

The classical Hardy-Littlewood-Sobolev inequality implies:

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|}\,dx\,dy \le C\|v\|^4_{L^{12/5}}.$$

Moreover, by using radial point-wise estimates, it is easy to prove that for $v \in E_r$,

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|}\,dx\,dy \;\le\; C\left(\int_{\mathbb{R}^3} v(x)^2|x|^{-\frac{1}{2}}\,dx\right)^2.$$

However, we need lower bounds for the Coulomb energy!

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$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|}\,dx\,dy \overset{?}{\ge} C\left(\int_{\mathbb{R}^3} v(x)^2|x|^{-\frac{1}{2}}\,dx\right)^2.$$

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A lower bound for the Coulomb energy

Theorem

Let $N \in \mathbb{N}$, $q > 0$, $\alpha > 1/2$. There exists $c > 0$ such that for any $v : \mathbb{R}^N \to \mathbb{R}$ measurable function, we have:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v^2(x)v^2(y)}{|x-y|^q}\,dx\,dy \geq c\left(\int_{\mathbb{R}^N} \frac{v(x)^2}{|x|^{\frac{q}{2}}(1+|\log|x||)^{\alpha}}\,dx\right)^2. \qquad (5)$$

The logarithmic term is necessary; actually, if $\alpha < \frac{N-2}{2N}$, (5) is not true.

We think that such inequality could be useful in other frameworks. In our problem, it implies that

$$E \subset L^2(\mathbb{R}^3,\ |x|^{-\frac{1}{2}}(1+|\log|x||)^{-\alpha}\,dx)$$

continuously.

L^p embeddings for E_r

Theorem

- *$E_r \subset L^p(\mathbb{R}^3)$ for $p \in (18/7, 6]$, and is compact for $p \in (18/7, 6)$.*
- *The above inclusion is false for $p < 18/7$.*

Take $\gamma > 1/2$; then, $E_r \subset H^1_r(\mathbb{R}^3, V)$, where

$$H^1_r(\mathbb{R}^3, V) = D^{1,2}_r(\mathbb{R}^3) \cap L^2(\mathbb{R}^3,\ V(x)\,dx),\ \ V(x) = \frac{1}{1 + |x|^\gamma}.$$

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L^p inclusions of these spaces have been studied, and there holds:

$$H^1_r(\mathbb{R}^3, V) \subset L^p(\mathbb{R}^3) \text{ for } p \in \left[\frac{2(4+\gamma)}{4-\gamma}, 6\right].$$

J. Su, Z. Q. Wang and M. Willem, 2007.

L^p embeddings for E_r

Since $\gamma > 1/2$, we obtain the inclusion for $p > 18/7$. The compactness is obtained by using uniform decay estimates.

L^p embeddings for E_r

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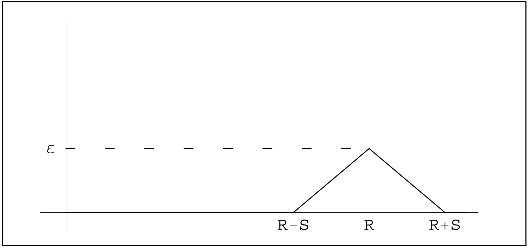
Moreover, let us define u_ε a radial function as depicted:



If we choose $R = \varepsilon^{-8/7}$, $S = \varepsilon^{-2/7}$ and make $\varepsilon \to 0$, we get:

$$\|u_\varepsilon\|_E = O(1) \ , \quad \int_{\mathbb{R}^3} u_\varepsilon^p \sim \varepsilon^{p-\frac{18}{7}} .$$

□

Back to the zero mass problem

We were interested in the problem:

$$-\Delta v + \left(v^2 \star \frac{1}{4\pi |x|} \right) v = |v|^{p-2} v. \qquad (6)$$

We can define the associated energy functional $J : E_r \to \mathbb{R}$,

$$J(v) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{v^2(x) v^2(y)}{|x-y|} \, dx \, dy - \frac{1}{p} \int_{\mathbb{R}^3} |v|^p \, dx,$$

and its critical points correspond to solutions of (6).

Theorem

For any $p \in (18/7, 6]$, J is well-defined and C^1. Moreover, if $p \in (18/7, 3)$, J is coercive and attains its infimum, which is negative.

Behavior of radial minimizers

$$\begin{cases} -\Delta u + u + \lambda \phi u = |u|^{p-2} u, \\ -\Delta \phi = u^2. \end{cases}$$

Theorem

Suppose that $p \in (18/7, 3)$, $\lambda_n \to 0$ *and* u_n *be a minimizer of* $I_{\lambda_n}|_{H^1_r}$.

Define $\varepsilon_n = \lambda_n^{\frac{p-2}{4(3-p)}} \to 0$ *and* v_n *by:*

$$u_n = \varepsilon_n^{-\frac{2}{p-2}} v_n \left(\frac{x}{\varepsilon_n} \right).$$

Then, $v_n \to v$ *in* E *(up to a subsequence) where* v *is a minimizer of* J.

Thank you for your attention!